Infinite slabs and other weird plane symmetric space-times with constant positive density

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We present the exact solution of Einstein's equation corresponding to a static and plane symmetric distribution of matter with constant positive density located below $z = 0$. This solution depends essentially on two constants: the density ρ and a parameter κ. We show that this space-time finishes down below at an inner singularity at finite depth. We match this solution to the vacuum one and compute the external gravitational field in terms of slab's parameters. Depending on the value of κ, these slabs can be attractive, repulsive or neutral. In the first case, the space-time also finishes up above at another singularity. In the other cases, they turn out to be semi-infinite and asymptotically flat when $z \to \infty$.

We also find solutions consisting of joining an attractive slab and a repulsive one, and two neutral ones. We also discuss how to assemble a "gravitational capacitor" by inserting a slice of vacuum between two such slabs.

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I. INTRODUCTION

Due to the complexity of Einstein's field equations, one cannot find exact solutions except in spaces of rather high symmetry, but very often with no direct physical application. Nevertheless, exact solutions can give an idea of the qualitative features that could arise in General Relativity, and so, of possible properties of realistic solutions of the field equations.

In this paper we want to illustrate some curious features of gravitation by means of a simple solution: the gravitational field of a static plane symmetric relativistic perfect incompressible fluid with positive density located below $z = 0$. Because of the symmetry required, the exterior gravitational field turns out to be Taub's plane vacuum solution [1]. The internal solution was also found by Taub [2].

Here we match both solutions, this corresponds to the plane symmetric counterpart of the Schwarzschild solution for a sphere of constant density [3].

The solutions turn out to be attractive, repulsive or neutral depending on the value of a parameter. These space-times present some somehow astonishing properties without counterpart in Newtonian gravitation:

Attractive solutions finish high above at an empty (free of matter) repelling boundary where space-time curvature diverges. These singularities are not the sources of the fields, but they arise owing to the attraction of distant matter as pointed out in [4].

Repulsive slabs explicitly show how negative but finite pressure can dominate the attraction of the matter.

We also consider matching two different internal solutions.

In Sec. II we present a simple and complete derivation of Taub's interior solution. In Sec. III we make a detailed study of it. In Sec. IV we discuss how solutions can be matched.

Throughout this paper, we adopt the convention in which the space-time metric has signature $(-\ +\ +\ +)$, the system of units in which the speed of light $c = 1$, Newton's gravitational constant $G = 1$ and g denotes gravitational field and not the determinant of the metric.

II. THE TAUB INTERIOR SOLUTION

In this section we consider the solution of Einstein's equation corresponding to a static and plane symmetric distribution of matter with constant positive density and plane symmetry. That is, it must be invariant under

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translations in the plane and under rotations around its normal. The matter we shall consider is a perfect fluid satisfying the equation of state $\rho = \rho_0$, where ρ_0 is an arbitrary positive constant. The stress-energy tensor is

$$T_{ab} = (\rho + p)\, u_a u_b + p\, g_{ab}\,, \tag{1}$$

where u^a is the velocity of fluid elements.

Due to the plane symmetry and staticity, following [1] we can find coordinates (t, x, y, z) such that

$$ds^2 = -\mathcal{G}(z)^2\; dt^2 + e^{2V(z)}\left(dx^2 + dy^2\right) + dz^2\,. \tag{2}$$

That is, it is the more general metric admitting the Killing vectors ∂_x, ∂_y, $x\partial_y - y\partial_x$ and ∂_t.

The non identically vanishing components of the Einstein tensor are

$$G_{tt} = -\,\mathcal{G}^2\left(2\,V'' + 3\,V'^2\right) \tag{3}$$

$$G_{xx} = G_{yy} = e^{2V}\left(\mathcal{G}''/\mathcal{G} + \mathcal{G}'/\mathcal{G}\;V' + V'' + V'^2\right)\,, \tag{4}$$

$$G_{zz} = V'\,(2\;\mathcal{G}'/\mathcal{G} + V')\,, \tag{5}$$

where a prime ($'$) denotes differentiation with respect to z.

On the other hand, since the fluid must be static, $u_a = (\mathcal{G}, 0, 0, 0)$, so

$$T_{ab} = \mathrm{diag}\left(\rho\,\mathcal{G}^2, p\,e^{2V}, p\,e^{2V}, p\right)\,, \tag{6}$$

where p depends only on the z-coordinante. Thus, Einstein's equations, i.e., $G_{ab} = 8\pi T_{ab}$, are

$$2\,V'' + 3\,V'^2 = -8\pi\rho\,, \tag{7}$$

$$\mathcal{G}''/\mathcal{G} + \mathcal{G}'/\mathcal{G}\;V' + V'' + V'^2 = 8\pi p\,, \tag{8}$$

$$V'\,(2\;\mathcal{G}'/\mathcal{G} + V') = 8\pi p\,. \tag{9}$$

Moreover, $\nabla_a T^{ab} = 0$ yields

$$p' = -(\rho + p)\,\mathcal{G}'/\mathcal{G}\,. \tag{10}$$

Of course, due to Bianchi's identities equations, (7), (8), (9) and (10) are not independent, so we shall here use only (7), (9), and (10).

Since ρ is constant, from (10) we readily find

$$p = C_p/\mathcal{G}(z) - \rho, \tag{11}$$

where C_p is an arbitrary constant.

By setting $W(z) = e^{3V(z)/2}$, we can write (7) as $W'' = -6\pi\rho W$, and its general solution can be written as

$$W(z) = C_1\,\sin\,(\sqrt{6\pi\rho}\;z + C_2), \tag{12}$$

where C_1 and C_2 are arbitrary constants. Therefore, we have

$$V(z) = \ln\left(C_1\,\sin\,(\sqrt{6\pi\rho}\;z + C_2)\right)^{\frac{2}{3}}\,. \tag{13}$$

Now, by replacing (11) into (9) we get the first order linear differential equation which $\mathcal{G}(z)$ obeys

$$\mathcal{G}' = -\left(\frac{4\pi\rho}{V'} + \frac{V'}{2}\right)\mathcal{G} + \frac{4\pi C_p}{V'} \tag{14}$$

$$= -\sqrt{6\pi\rho}\left(\tan u + \frac{1}{3}\cot u\right)\mathcal{G} + \sqrt{6\pi\rho}\;\frac{C_p}{\rho}\tan u\,, \tag{15}$$

where $u = \sqrt{6\pi\rho}\;z + C_2$, and in the last step we have made use of (13). The general solution of (14) can be written as

$$\begin{aligned}\mathcal{G} &= \frac{\cos u}{(\sin u)^{1/3}}\left(\text{const.} + \frac{C_p}{\rho}\int\frac{(\sin u)^{4/3}}{(\cos u)^2}du\right)\\ &= C_3\,\frac{\cos u}{(\sin u)^{1/3}} + \frac{3C_p}{7\rho}\,\sin^2 u\;{}_2F_1\!\left(1, \frac{2}{3}; \frac{13}{6}; \sin^2 u\right),\end{aligned} \tag{16}$$

where C_3 is another arbitrary constant, and ${}_2F_1(a,b;c;z)$ is the Gauss hypergeometric function (see for example [7]).

Therefore, the line element (2) becomes

$$ds^2 = -\mathcal{G}(z)^2\,dt^2 + (C_1\,\sin u)^{\frac{4}{3}}\left(dx^2+dy^2\right)+dz^2, \tag{17}$$

where $\mathcal{G}(z)$ is given in (16) and $u=\sqrt{6\pi\rho}\,z+C_2$. Thus, the solution contains five arbitrary constants: ρ, C_p, C_1, C_2, and C_3. The range of the coordinate z depends on the value of these constants.

For $\rho>0$, $C_p>0$ and $C_3>0$, this solution was found by Taub [2]. When $C_p=0$, it is clear from (11) that $p(z)=-\rho$, and the solution (17) turns out to be a vacuum solution with a cosmological constant $\Lambda=8\pi\rho$ [5]. For $C_p=0$ and $\rho\to 0$, by an appropriate choice of the constants we can readily see that (17) becomes

$$ds^2 = -(1-3gz)^{-\frac{2}{3}}\,dt^2 + (1-3gz)^{\frac{4}{3}}\left(dx^2+dy^2\right)+dz^2,$$

$$-\infty<t<\infty,\quad -\infty<x<\infty,\quad -\infty<y<\infty,\quad 0<1-3gz<\infty\,, \tag{18}$$

where g is an arbitrary constant. In (18) the coordinates have been chosen in such a way that it describes a homogeneous gravitational field g pointing in the negative z-direction in a neighborhood of $z=0$. The metric (18) is Taubs's vacuum plane solution [1] in the coordinates used in [4].

Nevertheless, the solution (17) has a wider range of validity. For *abnormal* matter some interesting solutions also arise, but the complete analysis turns out to be somehow involved. So, for the sake of clarity, we shall confine our attention to positive values of ρ and $C_p\neq 0$, leaving the complete study to a forthcoming publication [6].

Notice that the metric (17) has a space-time curvature singularity when $u=0$, since straightforward computation of the scalar quadratic in the Riemann tensor yields

$$\begin{aligned} R_{abcd}R^{abcd} &= 4\left(\mathcal{G}''^2+2\,\mathcal{G}'^2\,V'^2\right)/\mathcal{G}^2+4\left(2\,V''^2+4\,V''V'^2+3\,V'^4\right)\\ &= \frac{256}{3}\,\pi^2\rho^2\left(2+\sin^{-4}u+\frac{3}{4}\left(\frac{p}{\rho}+1\right)\left(\frac{3p}{\rho}-1\right)\right), \end{aligned} \tag{19}$$

so $R_{abcd}R^{abcd}\to\infty$ when $u\to 0$.

III. THE PROPERTIES OF THE FUNCTION $\mathcal{G}(z)$

In this section we shall study in detail the properties of the solution for the case $\rho>0$ and $C_p\neq 0$.

Now, it is clear from (7), (8), (9) and (10) that field equations are invariant under the transformation $z\to\pm z+z_0$, i.e., z-translations and mirror reflections across any plane z=const. Thus, if $\{\mathcal{G}(z),V(z),p(z)\}$ is a solution $\{\mathcal{G}(\pm z+z_0),V(\pm z+z_0),p(\pm z+z_0)\}$ is another one, where z_0 is an arbitrary constant. Therefore, taking into account that $u=\sqrt{6\pi\rho}\,z+C_2$, without loss of generality the consideration of the case $0<u<\pi/2$ shall suffice.

By an appropriate rescaling of the coordinates $\{t,x,y\}$, without loss of generality, we can write the metric (17) as

$$\begin{gathered} ds^2 = -\mathcal{G}(z)^2\,dt^2+\sin^{\frac{4}{3}}u\,\left(dx^2+dy^2\right)+dz^2,\\ -\infty<t<\infty,\quad -\infty<x<\infty,\quad -\infty<y<\infty,\quad 0<u=\sqrt{6\pi\rho}\;z+C_2\leq\pi/2, \end{gathered} \tag{20}$$

and (16) as

$$\mathcal{G}(z)=\frac{\kappa C_p}{\rho}\,\frac{\cos u}{\sin^{1/3}u}+\frac{3C_p}{7\rho}\,\sin^2 u\;\,{}_2F_1\!\Big(1,\frac{2}{3};\frac{13}{6};\sin^2 u\Big), \tag{21}$$

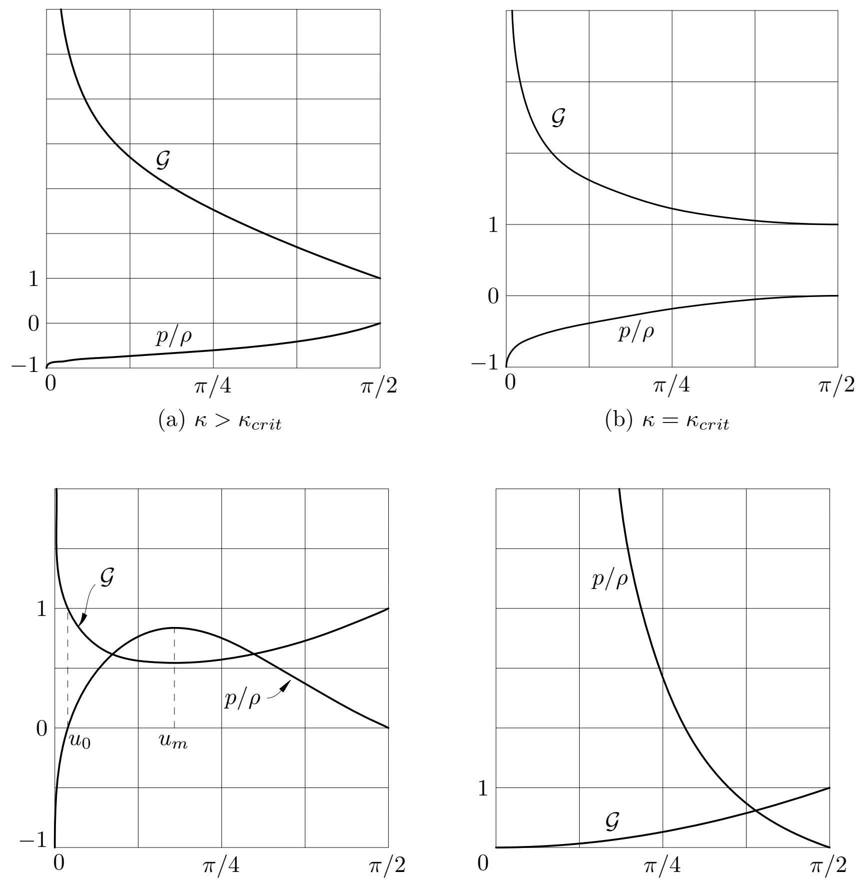
where κ is an arbitrary constant.

By replacing (21) into (11), we see that the pressure is independent of C_p. On the other hand, since $\mathcal{G}(z)$ appears squared in the metric, it suffices to consider $C_p>0$. Furthermore, rescaling again the coordinate t, we may set $C_p=\rho$. Thus, (21) becomes

$$\mathcal{G}(z)=G_\kappa(u)=\kappa\,\frac{\cos u}{\sin^{1/3}u}+\frac{3}{7}\,\sin^2 u\;\,{}_2F_1\!\Big(1,\frac{2}{3};\frac{13}{6};\sin^2 u\Big), \tag{22}$$

where $G_\kappa(u)$ was defined for future use, and we recall that $u=\sqrt{6\pi\rho}\,z+C_2$. Furthermore, (11) becomes

$$p(z)=\rho\left(1/\mathcal{G}(z)-1\right). \tag{23}$$



(a) $\kappa > \kappa_{crit}$ (b) $\kappa = \kappa_{crit}$

(c) $\kappa_{crit} > \kappa > 0$ (d) $\kappa = 0$

FIG. 1: $\mathcal{G}(z)$ and $p(z)$, as functions of u, for decreasing values of $\kappa \geq 0$.

Therefore the solution depends on two essential parameters, ρ and κ. We shall discuss in detail the properties of the functions $\mathcal{G}(z)$ and $p(z)$ depending on the value of the constant κ.

Now, the hypergeometric function in the last equation is a monotonically increasing continuous positive function of u for $0 \leq u \leq \pi/2$, since $c - a - b = 1/2 > 0$. Furthermore [8]

$$ {}_2F_1\Big(1, \frac{2}{3}; \frac{13}{6}; 0\Big) = 1, \quad \text{and} \quad {}_2F_1\Big(1, \frac{2}{3}; \frac{13}{6}; 1\Big) = \frac{7}{3}. \tag{24} $$

Therefore, we readily see from (22) that, no matter what the value of κ is, $\mathcal{G}(z)|_{u=\pi/2} = 1$ and we get then from (23) that $p(z)$ vanishes at $u = \pi/2$.

On the other hand, since

$$ \mathcal{G}(u) = \kappa\, u^{-1/3} + O(u^{5/3}) \qquad \text{as} \quad u \to 0\,, \tag{25} $$

$\mathcal{G}(z)|_{u=0} = 0$ if $\kappa = 0$, whereas it diverges if $\kappa \neq 0$.

For the sake of clarity, we shall analyze separately the cases $\kappa > 0$, $\kappa = 0$, and $\kappa < 0$.

1. $\kappa > 0$

In this case, it is clear from (22) that $\mathcal{G}(z)$ is positive definite when $0 < u \leq \pi/2$. On the other hand, from (8) and (9) we get

$$ \begin{aligned} \mathcal{G}'' = \mathcal{G}'V' - \mathcal{G}V'' &= -\Big(V'' + \frac{V'^2}{2} + 4\pi\rho\Big)\mathcal{G} + 4\pi C_p \\ &= V'^2\mathcal{G} + 4\pi C_p, \end{aligned} \tag{26} $$

where we have made use of (14) and (7). Then also $\mathcal{G}''$ is positive definite in $0 < u \leq \pi/2$, and so $\mathcal{G}'$ is a monotonically increasing continuous function of u in this interval.

Now, taking into account that $\mathcal{G}' = \partial_z \mathcal{G} = \sqrt{6\pi\rho}\, \partial_u \mathcal{G}$, a straightforward computation from (22) shows that

$$\mathcal{G}'(z) = -\frac{\kappa\sqrt{6\pi\rho}}{3} u^{-4/3} + O(u^{2/3}) \qquad \text{as } u \to 0, \tag{27}$$

and

$$\mathcal{G}'(z)|_{u=\pi/2} = \sqrt{6\pi\rho}\,(\kappa_{crit} - \kappa)\ , \tag{28}$$

where

$$\kappa_{crit} = \sqrt{\pi}\,\Gamma(7/6)/\Gamma(2/3) = 1.2143\ldots \tag{29}$$

If $\kappa \geq \kappa_{crit}$, $\mathcal{G}'$ is negative for small enough values of u and non-positive at $u = \pi/2$, hence $\mathcal{G}'$ is negative in $0 < u < \pi/2$, so $\mathcal{G}(z)$ is decreasing, and then $\mathcal{G}(z) > \mathcal{G}(z)|_{u=\pi/2} = 1$ in this interval (see Fig.1(a) and Fig.1(b)).

For $\kappa_{crit} > \kappa > 0$, $\mathcal{G}'$ is negative for sufficiently small values of u and positive at $\pi/2$. So there is one (and only one) value u_m where it vanishes. Clearly $\mathcal{G}(z)$ attains a local minimum there. Hence, there is one (and only one) value u_0 $(0 < u_0 < \pi/2)$ such that $\mathcal{G}(z)|_{u=u_0} = \mathcal{G}(z)|_{u=\pi/2} = 1$, and then $\mathcal{G}(z) < 1$ when $u_0 < u < \pi/2$ (see Fig.1(c)).

Since $\mathcal{G}(z) > 0$, it is clear from (23) that $p(z) > 0$ if $\mathcal{G}(z) < 1$, and $p(z)$ reaches a maximum when $\mathcal{G}(z)$ attains a minimum.

Therefore, for $\kappa \geq \kappa_{crit}$, $p(z)$ is negative when $0 \leq u < \pi/2$ and increases monotonically from $-\rho$ to 0 (see Fig.1(a) and Fig.1(b)).

On the other hand, for $\kappa_{crit} > \kappa > 0$, $p(z)$ grows from $-\rho$ to a maximum positive value when $u = u_m$ where it starts to decrease and vanishes at $u = \pi/2$. Thus, $p(z)$ is negative when $0 < u < u_0$ and positive when $u_0 < u < \pi/2$ (see Fig.1(c)).

2. $\kappa = 0$

In this case, it is clear from (22) that $\mathcal{G}$ monotonically increases with u from 0 to $\mathcal{G}(z)|_{u=\pi/2} = 1$. Therefore, p is a monotonically decreasing positive continuous function of u in $0 < u < \pi/2$ (see Fig.1(d)). Furthermore, at $u = 0$ it diverges, since

$$p(z) \sim \frac{7\rho}{3} u^{-2} \to +\infty \qquad \text{as } u \to 0. \tag{30}$$

3. $\kappa < 0$

In this case, we see from (27) that $\mathcal{G}'$ is positive when u takes small enough values, and from (28) that it is also positive when u is near to $\pi/2$.

Now, suppose that $\mathcal{G}'(z)$ attains a local minimum when $u = u_1$ $(0 < u_1 < \pi/2)$, then $\mathcal{G}''(z)|_{u=u_1} = 0$. Hence, we get from (26) that $\mathcal{G}(z)|_{u=u_1} < 0$, and taking into account that $V'(z)|_{u=u_1} = 2\sqrt{6\pi\rho}/3 \cot u_1 > 0$, we see from (14) that $\mathcal{G}'(z)|_{u=u_1} > 0$. Thus, we have shown that $\mathcal{G}'(z)$ is a continuous positive definite function when $0 < u \leq \pi/2$ if $\kappa < 0$.

Therefore, in this case, $\mathcal{G}(z)$ is a continuous function monotonically increasing with u when $0 < u \leq \pi/2$. Since it is negative for sufficiently small values of u and 1 when $u = \pi/2$ it must vanish at a unique value of z when $u = u_\kappa$ (say). Furthermore $\mathcal{G}(z) < 1$ when $0 < u < \pi/2$.

Clearly, we get from (22) that u_κ is given implicitly in terms of κ through

$$\kappa = -\frac{3}{7}\,\frac{\sin^{7/3} u_\kappa}{\cos u_\kappa}\ {}_2F_1\Big(1, \frac{2}{3}; \frac{13}{6}; \sin^2 u_\kappa\Big)\,. \tag{31}$$

We can readily see from (31) that u_κ is a monotonically decreasing function of κ in $-\infty < \kappa < 0$, and it tends to $\pi/2$ when $\kappa \to -\infty$ and to 0 when $\kappa \to 0^-$.

From (23), it is clear that $p(z)$ diverges when $u = u_\kappa$. Furthermore, (23) also shows that $p(z) < 0$ when $\mathcal{G}(z) < 0$, and taking into account that $\mathcal{G}(z) < 1$, that $p(z) > 0$ when $\mathcal{G}(z) > 0$. Therefore, $p(z)$ is negative when $0 < u < u_\kappa$ and positive when $u_\kappa < u < \pi/2$ (see Fig.2).

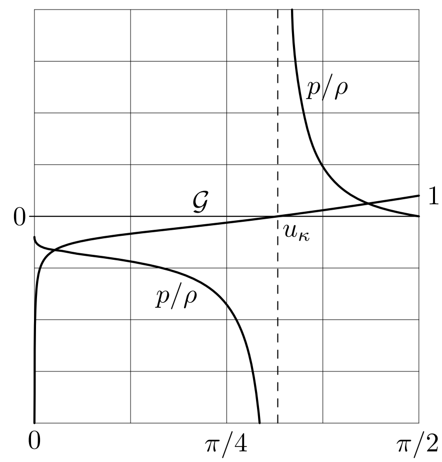


FIG. 2: $\mathcal{G}(z)$ and $p(z)$ as functions of u for $\kappa < 0$.

On the other hand, we see from (19) that, when κ is negative, another space-time curvature singularity arises at u_κ (besides the one at $u = 0$) since p diverges there.

Therefore, if κ is negative, the metric (20) describes two very different space-times:

(a) For $0 < u < u_\kappa$, the whole space-time is trapped between two singularities separated by a finite distance $\sqrt{6\pi\rho}\, u_\kappa$. This is a space-time full of a fluid with constant positive density ρ and negative pressure p monotonically decreasing with u, and $p(z)|_{u=0} = -\rho$ and $p(z) \to -\infty$ as $u \to u_\kappa$.

(b) For $u_\kappa < u < \pi/2$, the pressure is positive and monotonically decreasing with u, $p(z) \to \infty$ as $u \to u_\kappa$ and $p(z)|_{u=\pi/2} = 0$.

IV. MATCHING SOLUTIONS

In this section, we shall discuss matching the interior solution to the vacuum one, as well as joining two interior solutions facing each other.

Since the equations are invariant under z-translation, we can choose to match the solutions at $z = 0$ without losing generality. So we select $C_2 = \pi/2$, and then (22) becomes

$$\mathcal{G}(z) = G_\kappa(\sqrt{6\pi\rho}\, z + \pi/2) = -\kappa\, \frac{\sin(\sqrt{6\pi\rho}\, z)}{\cos^{1/3}(\sqrt{6\pi\rho}\, z)} + \frac{3}{7}\, \cos^2(\sqrt{6\pi\rho}\, z)\ \ {}_2F_1\Big(1, \frac{2}{3}; \frac{13}{6}; \cos^2(\sqrt{6\pi\rho}\, z)\Big)\,. \tag{32}$$

Therefore, the metric (20) reads

$$\begin{gathered} ds^2 = -G_\kappa(\sqrt{6\pi\rho}\, z + \pi/2)^2\, dt^2 + \cos^{\frac{4}{3}}(\sqrt{6\pi\rho}\, z)\ \left(dx^2 + dy^2\right) + dz^2, \\ -\infty < t < \infty, \quad -\infty < x < \infty, \quad -\infty < y < \infty, \quad -\sqrt{\pi/24\rho} < z \leq 0\,. \end{gathered} \tag{33}$$

Notice that it holds that $g_{tt}(0) = -\mathcal{G}(0)^2 = -1$, $g_{xx}(0) = g_{yy}(0) = 1$, and from (23) we get $p(0) = 0$.

Furthermore, we shall impose the continuity of $\partial_z g_{tt}$, and from (28) we have

$$\partial_z g_{tt}(0)|_{interior} = -2\, \mathcal{G}(0)\mathcal{G}'(0) = -2\sqrt{6\pi\rho}\, (\kappa_{crit} - \kappa)\ . \tag{34}$$

A. Matching solutions to vacuum solutions

Here, we shall consider the gravitational field of a planar symmetric distribution of matter with constant density ρ sitting below $z = 0$. This is the plane symmetric counterpart of Schwarzschild's solution for a sphere of incompressible fluid [3].

In this case, we see from (18) that the corresponding exterior solution for $z > 0$ is

$$ds^2 = -(1-3gz)^{-2/3}\,dt^2 + (1-3gz)^{4/3}\left(dx^2+dy^2\right) + dz^2,$$

$$-\infty < t < \infty, \quad -\infty < x < \infty, \quad -\infty < y < \infty, \quad 0 \leq z < \begin{cases} 1/3g & \text{if } g > 0 \\ \infty & \text{if } g \leq 0 \end{cases}, \tag{35}$$

which for $g > 0$ is the region finishing at the singularity of Taubs's vacuum plane solution, whereas it is the asymptotic flat tail of it for $g < 0$. It describes a homogeneous gravitacional field $-g$ in the vertical (i.e., z) direction [4].

Since $g_{tt}(0)|_{exterior} = -1$, $g_{xx}(0)|_{exterior} = g_{yy}(0)|_{exterior} = 1$ the continuity of the metric is assured, and concerning the derivative we have

$$\partial_z g_{tt}(z)|_{exterior} = -2g\,(1-3gz)^{-5/3}\,. \tag{36}$$

Therefore, by comparing with (34), we see that the continuity of $\partial_z g_{tt}$ yields

$$g = \sqrt{6\pi\rho}\,(\kappa_{crit} - \kappa)\;, \tag{37}$$

which relates the external gravitational field g with matter density ρ.

Thus, we can readily see from (37) that the slab is attractive if $\kappa < \kappa_{crit}$, and the whole space-time is trapped between two singularities: the inner one discussed in the previous section and the outer one at $z = 1/3g$.

If $\kappa \leq 0$, the range of the z-coordinate is $-(\pi/2 - u_\kappa)/\sqrt{6\pi\rho} < z < 1/3g$, where u_κ $(0 < u_\kappa < \pi/2)$ is given implicitly in terms of κ through (31). Inside the slab, the pressure is always positive, and it diverges deep below at the inner singularity (see Fig.2). Its depth is

$$d = (\pi/2 - u_\kappa)/\sqrt{6\pi\rho}\,. \tag{38}$$

By using (31), we can write κ in terms of d

$$\kappa = -\frac{3}{7}\,\frac{\cos^{7/3}(\sqrt{6\pi\rho}\,d)}{\sin(\sqrt{6\pi\rho}\,d)}\;\, {}_2F_1\!\left(1, \frac{2}{3}; \frac{13}{6}; \cos^2(\sqrt{6\pi\rho}\,d)\right). \tag{39}$$

From (37), (29) and (39), we can write the external gravitational field g in terms of the matter density ρ and the depth of the slab d

$$g = \sqrt{6\pi\rho}\left(\frac{\sqrt{\pi}\,\Gamma(7/6)}{\Gamma(2/3)} + \frac{3}{7}\,\frac{\cos^{7/3}(\sqrt{6\pi\rho}\,d)}{\sin(\sqrt{6\pi\rho}\,d)}\;\, {}_2F_1\!\left(1, \frac{2}{3}; \frac{13}{6}; \cos^2(\sqrt{6\pi\rho}\,d)\right)\right). \tag{40}$$

If $0 < \kappa < \kappa_{crit}$, the z-range is $-\sqrt{\pi/24\rho} < z < 1/3g$ and inside the slab, the pressure is always finite, but it is negative deep below and $p = -\rho$ at the inner singularity (see Fig.1(c)).

Therefore, when $\kappa < \kappa_{crit}$, we see how the attraction of the distant matter lying below $z = 0$ shrinks the space-time in such a way that it finishes high above at the outer empty singular boundary (at $z = 1/3g$).

It can readily be seen from (37) that, if $\kappa > \kappa_{crit}$, g is negative and the slab turns out to be repulsive. In this case, the space-time is semi-infinite and asymptotically flat when $z \to \infty$ (see [4]). Inside the slab, the pressure is always finite and negative, and $p = -\rho$ at the inner singularity (see Fig.1(a)).

If $\kappa = \kappa_{crit}$ it is *gravitationally neutral*, and the exterior is one half of Minkowski's space-time.

Two remarks are in order. First, notice that the maximum depth that a slab with constant density ρ can reach is $\sqrt{\pi/24\rho}$, being the counterpart of the well-known bound $M < 4R/9$ $(R < 1/\sqrt{3\pi\rho})$ which holds for spherical symmetry.

On the other hand, notice that the derivative of $g_{xx}(z) = g_{yy}(z)$ has a discontinuity at the surface $(z = 0)$ as it occurs with $\partial_r g_{rr}(r)$ in the Schwarzschild's case [3], since

$$\partial_z g_{xx}(0)|_{interior} = \partial_z g_{yy}(0)|_{interior} = -\frac{4\sqrt{6\pi\rho}}{3}\cos^{\frac{1}{3}}(\sqrt{6\pi\rho}\,z)\sin(\sqrt{6\pi\rho}\,z)|_{z=0} = 0\,, \tag{41}$$

but

$$\partial_z g_{xx}(0)|_{exterior} = \partial_z g_{yy}(0)|_{exterior} = -4g(1-3gz)^{1/3}|_{z=0} = -4g\,. \tag{42}$$

B. Matching two slabs

Now we consider two incompressible fluids joined at $z = 0$ where the pressure vanishes, the lower one having a density ρ and the upper having a density ρ'. Thus, the lower solution is given by (33). By means of the transformation $z \to -z$, $\rho \to \rho'$ and $\kappa \to \kappa'$ we get the upper one

$$ds^2 = -G_{\kappa'}(\pi/2 - \sqrt{6\pi\rho'}\, z)^2\, dt^2 + \cos^{\frac{4}{3}}(\sqrt{6\pi\rho'}\, z)\, \left(dx^2 + dy^2\right) + dz^2,$$
$$-\infty < t < \infty, \quad -\infty < x < \infty, \quad -\infty < y < \infty, \quad 0 \le z < \sqrt{\pi/24\rho'}\,. \tag{43}$$

From (33) and (43), we can readily see that $g_{tt}(z)$, $g_{xx}(z)$ and $\partial_z g_{xx}(z)$ are continuous at $z = 0$. Furthermore, from (34) we see that the continuity of $\partial_z g_{tt}$ requires

$$\sqrt{\rho}\,(\kappa_{crit} - \kappa) = -\sqrt{\rho'}\,(\kappa_{crit} - \kappa')\ . \tag{44}$$

Thus, if one solution has a κ greater than κ_{crit}, the other must have it smaller than κ_{crit}. Therefore, the joining is only possible between an attractive solution and a repulsive one, or between two neutral ones.

It is easy to see that we can also insert a slice of arbitrary thickness of the vacuum solution (18) between them, obtaining a full relativistic plane "gravitational capacitor". For example, we can trap a slice of Minkowski's space-time between two solutions with $\kappa = \kappa_{crit}$.

V. CONCLUDING REMARKS

We have done a detailed study of the exact solution of Einstein's equations corresponding to a static and plane symmetric distribution of matter with constant positive density. By matching this internal solution to the vacuum one, we showed that different situations arise depending on the value of a parameter κ. These simple space-times turn out to present some somehow astonishing properties.

For $\kappa < \kappa_{crit}$, the attraction of the distant matter shrinks the space-time in such a way that it finishes high above at an empty singular boundary as pointed out in [4]. This space-time also finishes down below at another singularity.

For $\kappa < 0$, we have explicitly computed the external gravitational field g in terms of the density ρ and the sickness d of the slab.

For $\kappa > \kappa_{crit}$, negative but finite pressure dominates the attraction of the matter and the slab turns out to be repulsive.

We showed that the maximal sickness that these slabs can have is $\sqrt{\pi/24\rho}$.

We have also discussed matching an attractive slab to a repulsive one, and two neutral ones. We also comment on how to assemble relativistic gravitational capacitors consisting of a slice of vacuum trapped between two of such slabs.

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